Imperial College
London

# Topic 9

## Reactance, Impedance and filter circuit

Professor Peter YK Cheung
Dyson School of Design Engineering

URL: www.ee.ic.ac.uk/pcheung/teaching/DE1_EE/
E-mail: p.cheung@imperial.ac.uk



In the last lecture, I introduced the capacitor as an electronic component that stores charges. I then consider the mechanism of charging and discharging a capacitor using a DC source and through a resistor.

In this lecture, we will consider how a capacitor behaves in a circuit when the source signal is not a dc, but an ac. To be specific, we will consider how capacitor affects sine or cosine wave signals.

We will also introduce a characteristic of a capacitor known as reaction, and its related quantity called impedance, which is similar to (but different from) resistance in a resistor.

## Steady State vs Transient

- An electronic circuit could be responding to either fixed DC signals (such as constant voltage or current sources) or fixed sine, cosine or repetitive signals. The response of the circuit is known as "**steady state response**".
- However, before a circuit reaches steady state, it generally goes through a period of sudden changes, such as being switched from OFF to ON.
- The response of the circuit to these sudden changes is referred to as the "**transient response**".
- If the stimulus to the electronic system is a step function (e.g. it goes from a low voltage level to a high voltage level), the response is known as "**step response**".



It is helpful to think of electronic systems in two different "**states**". If you drive a system with a constant source (such as a battery) or a periodic signal (such as a sine wave), eventually the voltage and current in the system to settle down to a state which will sort of last for ever! We call this the "**steady state**".

However if you suddenly connect the system to a battery (such as closing a switch) , then the system will take some time to adapt to this sudden change. During this period, we call the system to be in "**transient state**".

The total response of a system is a combination of the response to these two states added together.

## Initial and Final values

- RC or CR circuits are called **first-order systems** because their behaviours are determined with a first-order differential equation.
- We can generalize first-order system transient responses in terms of two exponentials:

$$v = V_f + (V_i - V_f) \times e^{-\frac{t}{\tau}} \qquad i = I_f + (I_i - I_f) \times e^{-\frac{t}{\tau}}$$

where $V_i$ and $I_i$ are the **initial values** of the voltage and current, and $V_f$ and $I_f$ are the **final values** of the voltage and current.

- The first terms in these expressions are the **steady-state responses** of the circuit when t ➔∞.
- The second terms in these expressions are the **transient responses** of the circuit.
- Together they provide the voltage and current values instantaneously and when t is long. These are the **total responses** of the circuits.

PYKC 21 May 2020 DE1.3 - Electronics 1 Topic 9 - Slide 3

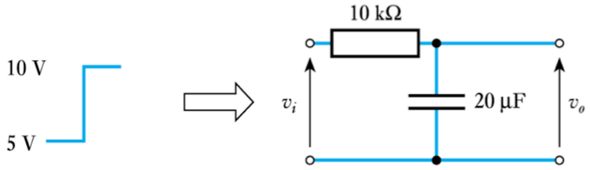
We can combine both cases into a general theorem relating initial and final values. This only applies to systems that are FIRST-ORDER and linear.

The exponential equations have $V_i$ and $I_i$, which are the initial values, and $V_f$ and $I_f$, which are final values. The final values are the steady state values. The exponential governs the transient response of the system.

When we add the steady state to the transient, we get the **total response** of the system.

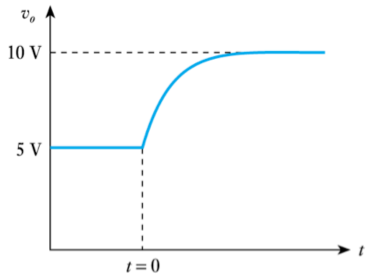
## An Example

- The input voltage to the following RC network undergoes a step change from 5 V to 10 V at time t = 0. Derive an expression for the resulting output voltage.



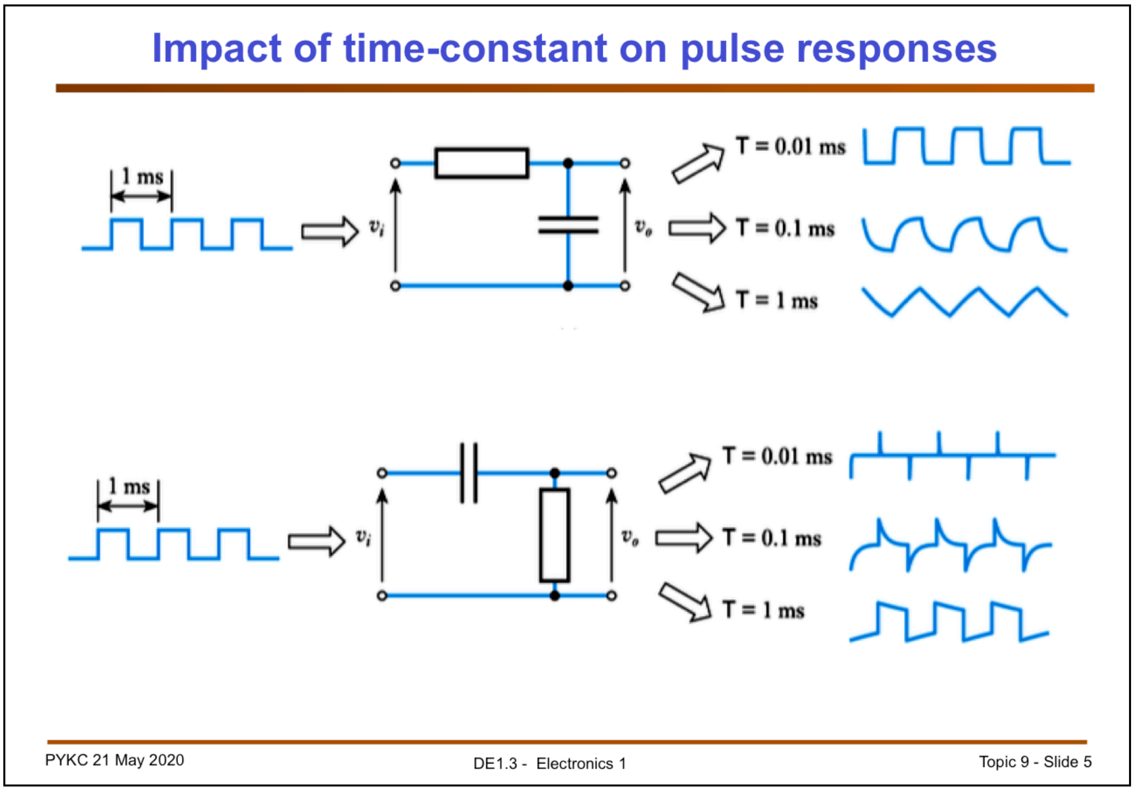

- Here the initial value is 5 V and the final value is 10 V. The time constant of the circuit equals RC = 10 × $10^3$ ×20 × $10^{-6}$ = 0.2s. Therefore, from above, for t ≥ 0.

$$\begin{aligned} v &= V_f + (V_i - V_f) \times e^{-t/T} \\ &= 10 + (5 - 10) \times e^{-t/0.2} \\ &= 10 - 5e^{-t/0.2} \text{ volts} \end{aligned}$$






Here is an example with a RC circuit drive by a step function from 5V to 10V.




Time constant has significant impact on shape of pulses and pulse train passing through a system. This is important to appreciate because digital signals are affected by RC effects in real electronic circuits. You will be experimenting with these circuits when they are driven by digital (CLOCK) signals in Lab 2.

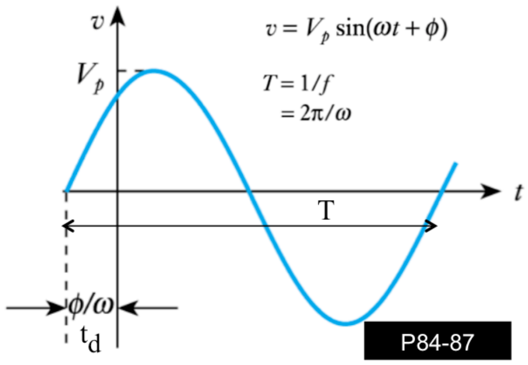
## Sine and Cosine waves

- We have considered sine wave signals earlier in lectures and labs. We know a sinusoidal signal is given by the equation:

$$v(t) = V_p \sin(2\pi f t + \phi)$$

where $V_p$ is the peak voltage
$f$ is the frequency in Hz
Φ is the phase angle, either in radians or in degrees

- We often use ω, the angular frequency in rad/sec, instead of *f*, and ω = 2π *f*
- If Φ is in radians, then the time shift $t_d$ is given by Φ/ω.
- Remember that period T = 1/f and one cycle of a sine wave corresponds to a phase angle of 2π radians or 360 degrees.






Let us revisit sine and cosine waves. A sine wave can be completely defined with three parameters Vp, the peak voltage (or amplitude), its frequency w in radians/second or f in cycles/second (Hz), and the phase angle Φ.

Cosine waves are the same as sine waves, except that cosine wave is 90 degrees or π/2 radians advance in phase. That is:

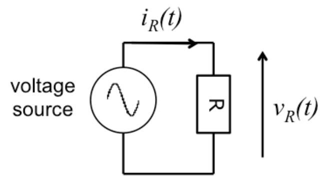
$$v(t) = V_p \sin(2\pi f t + \phi) \;\; = V_p \cos(2\pi f t + \phi + \frac{\pi}{2})$$

It is worth remembering that one cycle of a sine or cosine wave has a phase angle value of 2π radians or 360 degrees.

## Sine wave through a resistor and a capacitor

- Consider the resistor circuit driven by a sinusoidal voltage source:

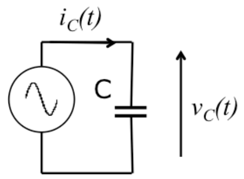
$$v_R(t) = V_{pk} \sin \omega t$$




- Using Ohm's law, we have:

$$i_R(t) = v_R(t) / R = \frac{1}{R} V_{pk} \sin \omega t$$

- Now consider a capacitor driven by the same signal $v_R(t)$.
- Capacitor equation is: $i_C(t) = C \dfrac{dv_C(t)}{dt}$




- Hence: $v_C(t) = V_{pk} \sin \omega t$

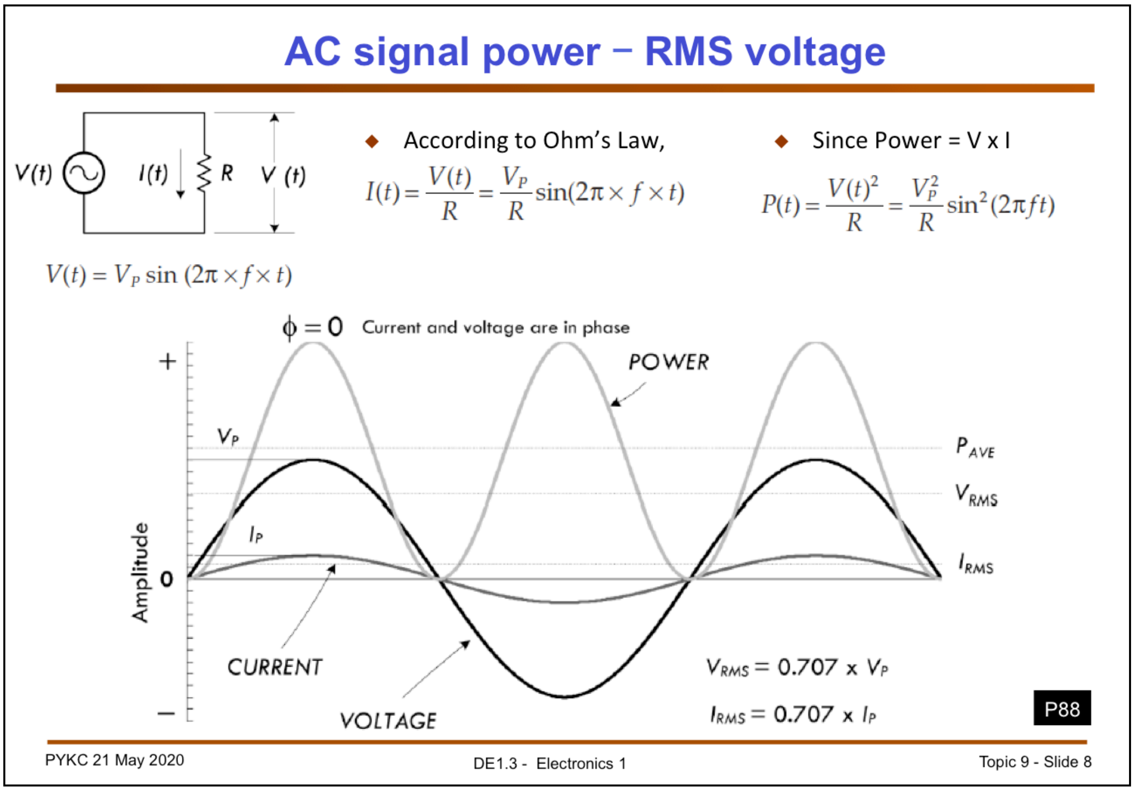
$$i_C(t) = C \frac{d}{dt}(V_{pk} \sin \omega t) = \omega C \times V_{pk} \cos \omega t$$

PYKC 21 May 2020 DE1.3 - Electronics 1 Topic 9 - Slide 7

Resistors obey Ohm's Law – the ratio of voltage to current $V_R/I_R$ through a resistor is a constant no matter what is the frequency of the source signal.

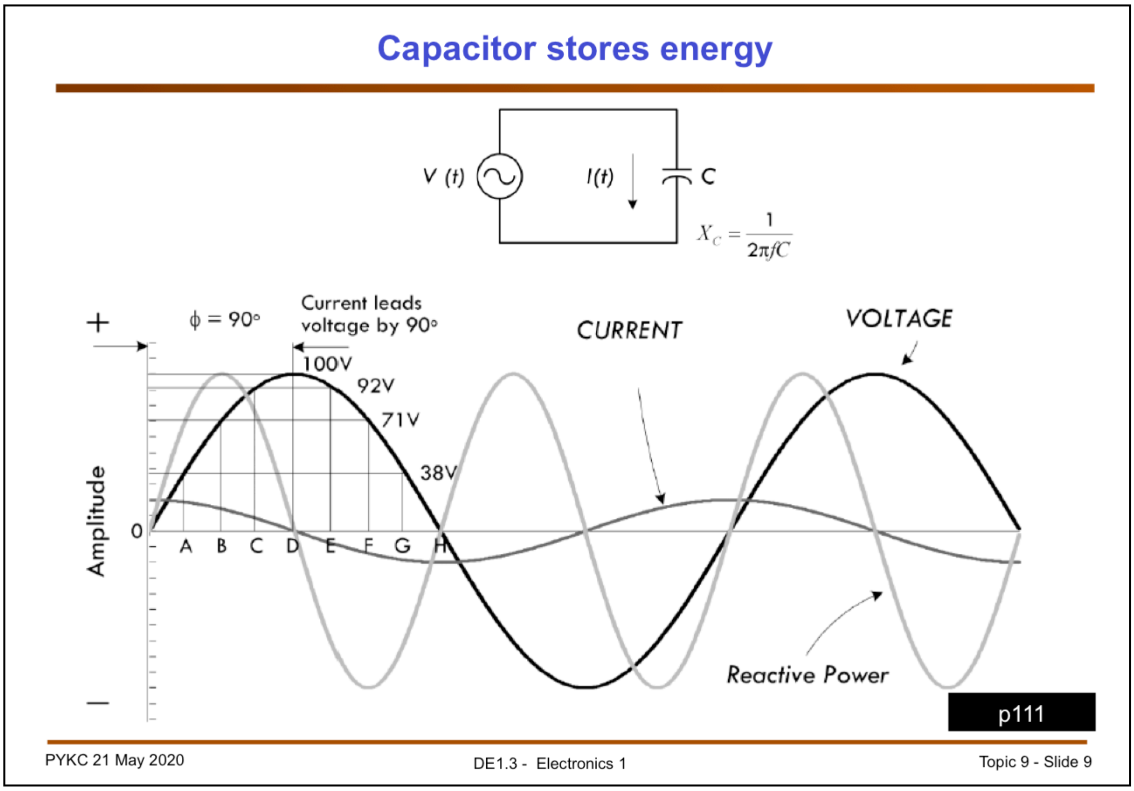
This is not true with a capacitor. As shown in the slide, the ratio $V_C/I_C$ when driven by a sine wave source results in a current that is a cosine wave.

It can be seen that the ratio of $V_C/I_C$ is dependent on the signal frequency $\omega$. This ratio (capacitor's version of resistance) is inversely proportional to the signal frequency. When $\omega$ = 0, the ratio is infinite; when $\omega$ is very large (infinity), the ratio approaches zero.




This is a partial repeat of Topic 6 slide 6. Remember, with a sine wave voltage source applied to a resistor, the current is also a sine wave in phase with the voltage. Therefore when multiply the two together to get the power dissipation, P = V x I, we have the function that is entirely positive.

The physical interpretation of this observation is that although a sine wave has positive and negative part in one cycle, the power is always positive. Therefore the resistor dissipates (or consumes) energy throughout the entire cycle of the signal.




Now consider the capacitor voltage and current plotted on the same axis in time. The voltage Vc(t) is clearly a sine wave. The current Ic(t) is a cosine wave.

In a resistor, the power dissipated is the produce of V and I. So we should ask the question, what is the power dissipated by the capacitor?

Let us calculate the instantaneous power "dissipated" by the capacitor by finding the produce of Vc(t) and Ic(t). This is plotted as the reactive power curve in the slide.

Unlike the power curve we saw for a resistor, the power curve for a capacitor spends half the cycle in the +ve part, and half of the cycle in the −ve part of the power y-axis. That means on half the time, the capacitor is using energy, and on the other half of the time, it is giving the energy back! On average, there is zero energy dissipated. The capacitor simply stores the energy and return it later.

This is unlike a resistor, which only DISSIPATES energy. That why the name given the ratio $V_C/I_C$ is **reactance** (as suppose to resistance). The power curve is known as reactive power.

## V/I relationships for R and C (peak only)

- Let us ignore phase angle for the moment, and compute **peak voltage** and **peak current** in both cases.
- Resistor (resistance):

$$\frac{v_R(t)_{\max}}{i_R(t)_{\max}} = \frac{(\mathrm{V}_{pk}\sin\omega t)_{\max}}{(V_{pk}\sin\omega t / R)_{\max}} = R$$

- Capacitor (reactance):

$$X_C = \frac{v_C(t)_{\max}}{i_C(t)_{\max}} = \frac{(V_{pk}\sin\omega t)_{\max}}{\omega C(V_{pk}\cos\omega t)_{\max}} = \frac{1}{\omega C}$$



Let us for the moment just consider ratio of the **peak magnitudes** of the voltage and the current in the three cases.

The ratio is simply R for resistor.

The ratio is 1/ωC for capacitor.

## Reactance of Capacitor

- The ratio of voltage to current is a measure of how the component opposes the flow of electricity
- In a resistor, this ratio is the **resistance**
- In capacitors it is termed its **reactance**
- Reactance is given the symbol *X*. Therefore:

$$\text{Reactance of a capacitor, } X_C = \frac{1}{\omega C}$$

- Units of reactance is ohms, same as resistance.
- It can be used in much the same way as resistance: $V = I X_C$ $V = I X_L$
- Example: A sinusoidal voltage of 5 V peak and 100 Hz is applied across an inductor of 25 mH. What will be the peak current?

$$X_L = \omega L = 2\pi f L = 2 \times \pi \times 100 \times 25 \times 10^{-3} = 15.7\Omega$$

$$\text{Therefore} \quad I_L = \frac{V_L}{X_L} = \frac{5}{15.7} = 0.318A \ \text{(peak)}$$

PYKC 21 May 2020 DE1.3 - Electronics 1 Topic 9 - Slide 11

For capacitors and inductors, this ratio of peak voltage over peak current is frequency dependent. They are called **reactance**.

Both resistance and reactance are measures of how the components oppose the flow of current. The unit of reactance is the same as that of resistance – in ohms.

We use the symbol X to represent reactance here.

## Impedance – reactance + phase

- However, remember that peak voltage and peak current in a capacitor happen at different time

$$v_C(t) = V_{pk} \sin \omega t \qquad i_C(t) = \omega C \times V_{pk} \cos \omega t$$

- Furthermore, for sine signals, capacitor current always **LEADS** capacitor voltage by 90 degrees or $\pi/2$
- To account of the constant phase difference between the two peaks, we define the ratio of the amplitude of capacitor voltage / capacitor current as a complex quantity known as **impedance**, such that:

**Impedance:** $v_c(t) / i_C(t) = 1 / j\omega C$

- The ratio of voltage to current in a capacitor is now a complex number
- The use of complex number allows us to treat capacitors in a similar way to resistor – all analysis we used for resistors also works here, as long as we use complex number in our calculations

PYKC 21 May 2020 DE1.3 - Electronics 1 Topic 9 - Slide 12

When we define reactance, we only consider peak voltage and current in a component because we only use the maximum (or peak) voltage and current in the calculation. However, the peak current and voltage in a capacitor happens at different times. In fact, we have seen that the current in a capacitor (when a sine wave is applied) always LEADS the voltage by 90 degrees. Therefore, how can we use reactance in nodal analysis? The answer is to use complex number representation.

We define here a new quantity, the IMPEDANCE of a capacitor as the reactance, but add the complex variable I in the denominator. So the impedance Zc is:

$$Z_C = \frac{1}{i\omega C}$$

However using i is very confusion for electrical engineers. The symbol “i” also represents current. Therefore we electrical engineers, unlike mathematicians, always use j to represent the imaginary part of a complex number.

$$Z_C = \frac{1}{j\omega C}$$

## Gain of a Two-port Networks

- While the properties of a pure ***resistance*** are not affected by the frequency of the signal concerned, this is not true of ***reactive*** components.
- We will start with a few basic concepts and then look at the characteristics of simple combinations of resistors and capacitors.
- A two-port network has two ports: an input port, and an output port.
- We can define voltages and currents at the input and output as shown here.
- Then:

$$\text{voltage gain } (A_v) = \frac{V_O}{V_i}$$

$$\text{current gain } (A_i) = \frac{I_O}{I_i}$$

$$\text{power gain } (A_p) = \frac{P_O}{P_i}$$




Reactive components are things like capacitors and inductors. Their reactance (equivalent to resistance in resistors) are frequency dependent. This frequency dependency turns out to be very useful in building interesting circuits, e.g. filters which provide frequency selectivity.

Before we look at these circuits, I want to introduce to idea of two port network and gain.

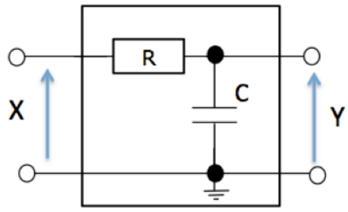
A two-port network as an input port to which we apply stimulus $V_i$. There is an output port that provides a signal $V_O$. The ratio $V_O/V_I$ is the voltage gain.

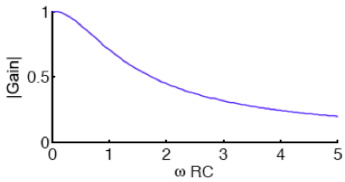
## Frequency Response

- If $x(t)$ is a sine wave, then $y(t)$ will also be a sine wave but with a different amplitude and phase shift. X is an input phasor and Y is the output phasor.
- The *gain* of the circuit is $\frac{Y}{X} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{j\omega RC + 1}$
- This is a complex function of $\omega$ so we plot separate graphs for:

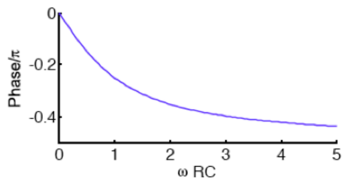
Magnitude: $\left|\frac{Y}{X}\right| = \frac{1}{|j\omega RC+1|} = \frac{1}{\sqrt{1+(\omega RC)^2}}$

Phase Shift: $\angle\left(\frac{Y}{X}\right) = -\angle\left(j\omega RC + 1\right) = -\arctan\left(\frac{\omega RC}{1}\right)$




Magnitude Response




Phase Response



Using complex number algebraic methods, we can easily work out the voltage gain of this simple RC network. Note that the gain equation is frequency dependent (i.e. it is a function of $\omega$).

The relationship between output Y and input X as a function of signal frequency $\omega$ is known as **frequency response**.

From the gain equation, we can compute the magnitude of the gain as a function of frequency. We can also plot the phase difference (output relative to input) as a function of frequency. The former is known as the magnitude (or amplitude) response. The latter is known as the phase response.

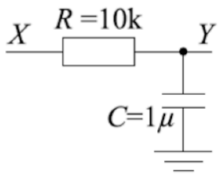
In the literature, graphs showing gain magnitude and phase vs frequency is also known as “**Bode diagrams**”. Don’t worry, it is just a name! It is more important to know what it means: it is a plot of gain vs frequency of a circuit.

On this course, we will mainly focus on the magnitude response, and **we will ignore the phase response most of the time**.

# Sine Wave Response

$RC = 10$ ms

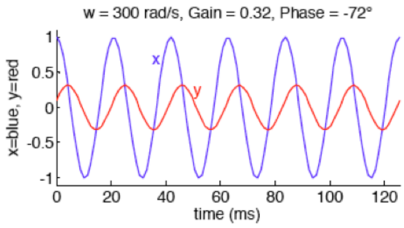
$$\frac{Y}{X} = \frac{1}{j\omega RC+1} = \frac{1}{0.01j\omega+1}$$

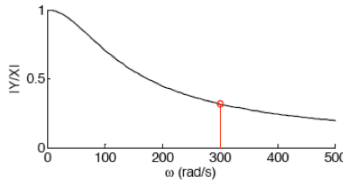


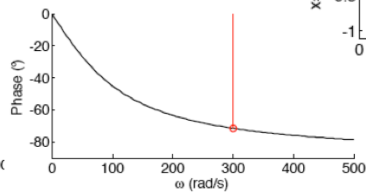

$$\omega = 50 \Rightarrow \frac{Y}{X} = 0.89\angle -27°$$

$$\omega = 100 \Rightarrow \frac{Y}{X} = 0.71\angle -45°$$

$$\omega = 300 \Rightarrow \frac{Y}{X} = 0.32\angle -72°$$



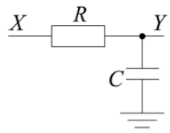

- The output, $y(t)$, lags the input, $x(t)$, by up to $90°$.



Here is a circuit with R = 10k and C = 1 microfarad, and how the gain magnitude and phase changes with frequency. Note that at w = 1/RC, the gain is 0.71 (or 1/√2). This will come up again later.

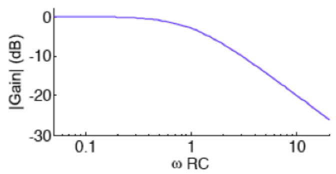
## Logarithmic axes

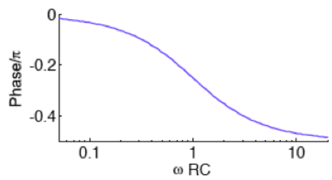
- We usually use logarithmic axes for frequency and gain (but not phase) because % differences are more significant than absolute differences.
- E.g. 5 kHz versus 5.005 kHz is less significant than 10Hz versus 15Hz even though both differences equal 5Hz.
- Logarithmic voltage ratios are specified in *decibels* (dB) = $20 \log_{10} |V_2 / V_1|$.




- Common voltage ratios:

| $\frac{|V_2|}{|V_1|}$ | 0.1 | 0.5 | $\sqrt{0.5}$ | 1 | $\sqrt{2}$ | 2 | 10 | 100 |
|---|---|---|---|---|---|---|---|---|
| dB | −20 | -6 | -3 | 0 | 3 | 6 | 20 | 40 |




Note that 0 does not exist on a log axis and so the starting point of the axis is arbitrary.

Note: $P \propto V^2 \Rightarrow$ decibel power ratios are given by $10 \log_{10} (P_2 / P_1)$

P204-206

PYKC 21 May 2020 DE1.3 - Electronics 1 Topic 9 - Slide 16

Percentage differences are often more important than absolute differences. Therefore it is often more revealing if we plot frequency and gain in **logarithmic scales**.

For voltages, we normally calculate the gain in terms of decibels (dB) as defined here.

Remember that dB is dimensionless – it is a scaling, not a unit.

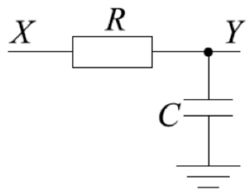
The definition of Gain in decibel is:

$$\boxed{\textit{decibels}\ (\mathrm{dB}) = 20 \log_{10} |V_2 / V_1|}$$

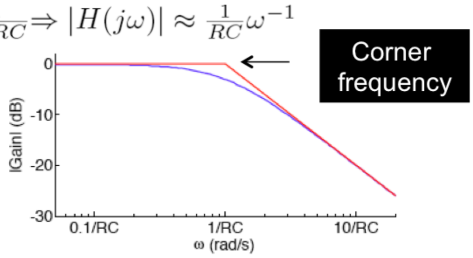
When expressing power gain, the formula is different – the constant before the log is 10x not 20x.

## Straight Line Approximations

- Key idea: $(aj\omega + b) \approx \begin{cases} aj\omega & \text{for } |a\omega| \gg |b| \\ b & \text{for } |a\omega| \ll |b| \end{cases}$




- Gain: $H(j\omega) = \frac{1}{j\omega RC+1}$
- Low frequencies: $(\omega \ll \frac{1}{RC})$: $H(j\omega) \approx 1 \Rightarrow |H(j\omega)| \approx 1$
- High frequencies: $(\omega \gg \frac{1}{RC})$: $H(j\omega) \approx \frac{1}{j\omega RC} \Rightarrow |H(j\omega)| \approx \frac{1}{RC}\omega^{-1}$
- Approximate the magnitude response as two straight lines intersecting at the corner frequency, ωc = 1/RC .




- At the corner frequency:

  (a) the gradient changes by −1 (= −6 dB/octave = −20 dB/decade).

  (b) $|H(j\omega_c)| = \left|\frac{1}{1+j}\right| = 1/\sqrt{2} = -3$ dB (worst-case error).

A linear factor $(aj\omega + b)$ has a corner frequency of $\omega_c = |b/a|$.



I now want to show you how to make informed guess to the magnitude response of a circuit from its gain equation without having to do any calculations. This gain equation is frequency dependent and is often written as H(jω). Since it is the ratio of output voltage to input voltage (or output current to input current), it is called a **TRANSFER FUNCTION**:

Y(jω) = H(jω) X(jω)

Now consider the function: H(jω) = a x jω + b.

If the magnitude |aω| is much small than |b|, |H(jω)| ➔ |b|.

If the magnitude |aω| is much larger than |b|, |H(jω)| ➔ |aω|.

Let us take the RC circuit as shown here. For low frequencies, the magnitude is 1 or 0 dB.

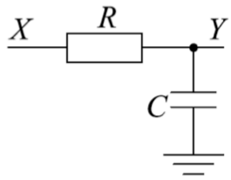
For high frequencies the magnitude drops linearly with ω (i.e. it is proportional to $\omega^{-1}$. So, in terms of dBs, over one decade (factor of 10), it falls by 20dB. The slope of the line at high frequency is therefore -20dB/ decade.

How high must the frequency be before it is called "high"? The intersection of these two line is at ω = 1/RC. This is known as the corner frequency.

If you use Hz for frequency, then f = 1/2π RC.

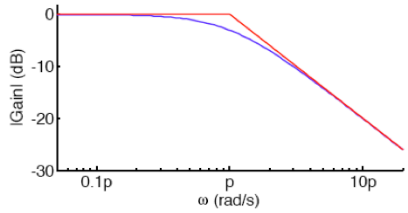
## $1^{st}$ Order Low Pass Filter

$$\frac{Y}{X} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{j\omega RC + 1} = \frac{1}{\frac{j\omega}{p} + 1}$$




- Corner frequency: $p = \left|\frac{b}{a}\right| = \frac{1}{RC}$
- Asymptotes: 1 and $\frac{p}{j\omega}$

  Very low $\omega$: Capacitor = open circuit
  Very high $\omega$: Capacitor = short circuit




- A low-pass filter because it allows low frequencies to pass but ***attenuates*** (makes smaller) high frequencies.
- The ***order*** of a filter: highest power of $j\omega$ in the denominator.
- Almost always equals the total number of L and/or C.

P210-213

PYKC 21 May 2020 DE1.3 - Electronics 1 Topic 9 - Slide 18

Let us take another look at the simple RC network. Remember that the impedance of a capacitor is inversely proportional to frequency. Therefore at low frequency, a capacitor appears as open-circuit. At high frequency, it appears as short-circuit.

Using the principle of voltage divider, this circuit will give you a low output at high frequency ($Z_C$ is small), and does not attenuate the signal at low frequency ($Z_c$ is large). We call this a low pass filter (LF).

The transfer function H(jω) = Y/X has only one jw term in the denominator. We call this first order filter.

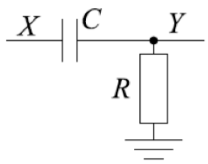
The order of a filter is the highest power of the (jω) term in the denominator of the transfer function (Gain function).

A first order filter will have a roll-off (i.e. rate of drop in gain) of -20dB/decade.

An nth order filter will have a roll-off of -20n dB/decade.

## High Pass Filter

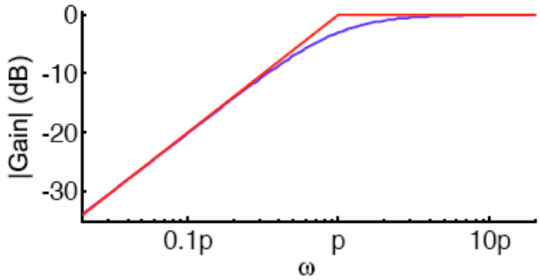
$$\frac{Y}{X} = \frac{R}{R+1/j\omega C} = \frac{j\omega RC}{j\omega RC+1}$$




- Corner frequency: $p = \frac{1}{RC}$
- Asymptotes: $j\omega RC$ and 1

Very low ω:
Capacitor = open circuit
Gain = 0

Very high ω:
Capacitor short circuit
Gain = 1




PYKC 21 May 2020 DE1.3 - Electronics 1 Topic 9 - Slide 19

If you swap the RC and to form a CR circuit as shown, we have a magnitude response where at high frequency, C appears to be short circuit and Y=X. However C blocks any low frequency and DC signals. Therefore we now have a high pass filter.

Again you can work out the straight line approximation with the gain equation (transfer function).